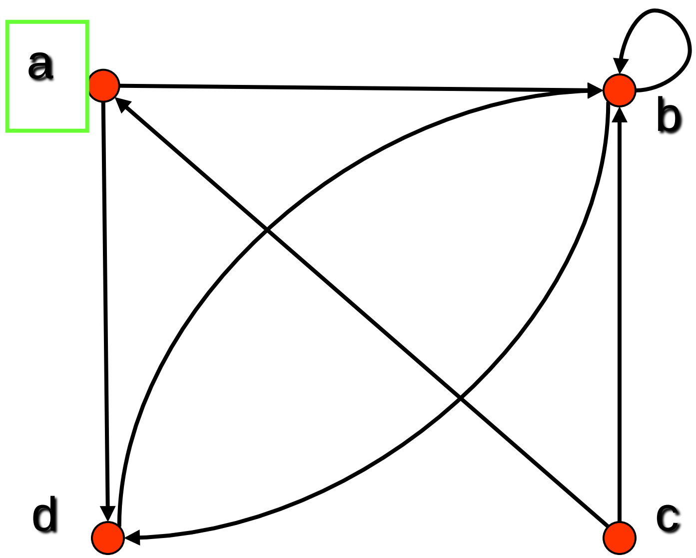
4. Relations and Digraphs

Representing Relations Using Digraphs

• **Example:** Display the digraph with V = {a, b, c, d}, E = {(a, b), (a, d), (b, b), (b, d), (c, a), (c, b), (d, b)}.



An edge of the form (b, b) is called a **loop**.

4.4 Properties of Relations

Reflexive Property of a Relation

Definition:

A relation *R* on a set *A* is called *reflexive* if (a, a) $\in$ *R* for every element $a \in A$.

Example 4.6-1 Consider the following relations on A={1,2,3,4}. Which of these relations are reflexive?

R_1 = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)}

Not reflexive because 3 $\in$ A but (3,3) $\notin$ R_1

R_2 = {(1, 1), (1, 2), (2, 1)}

Not reflexive because, say, 4 $\in$ A but (4, 4) $\notin$ R_2

R_3 = {(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)}

Reflexive

$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$

Not reflexive - (1, 1) ?

$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3),$
$(2, 4), (3, 3), (3, 4), (4, 4)\}$

Reflexive - Why ?

$R_6 = \{(3, 4)\}$

Not Reflexive - Why ?

Examples 1- c

(c) Let $A = \{1, 2, 3\}$, and let $R = \{(1, 1), (1, 2)\}$. Then R is not reflexive since $(2, 2) \notin R$ and $(3, 3) \notin R$. Also, R is not irreflexive, since $(1, 1) \in R$.

Symmetric Property of a Relation

Definitions:

A relation R on a set A is called *symmetric* if for all $a, b \in A$, $(a, b) \in R$ implies $(b, a) \in R$.

A relation R on a set A is called *antisymmetric* if for all $a, b \in A$,

$(a, b) \in R$ and $(b, a) \in R$ implies $a = b$.

Example 4.6-2 **Which of the relations are symmetric and which are antisymmetric?**

R_1 = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)}

Not symmetric - (3, 4) but there is no (4, 3)

Not antisymmetric - (1, 2) & (2, 1) but $1 \neq 2$

R_2 = {(1, 1), (1, 2), (2, 1)}

Symmetric

Not antisymmetric - (1, 2) & (2, 1) but $1 \neq 2$

R_3 = {(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)}

Symmetric

Not antisymmetric - (1, 4) & (4, 1) but $1 \neq 4$

$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$

Not symmetric - (2, 1) but no (1, 2)

Antisymmetric

$R_5 = \{ (1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4) \}$

Not symmetric - (1, 3) but no (3, 1)

Antisymmetric

$R_6 = \{(3, 4)\}$

Not symmetric - (3, 4) but no (4, 3)

Antisymmetric

Examples 4

Example 4 Let $A = \{1, 2, 3, 4\}$ and let

$$R = \{(1, 2), (2, 2), (3, 4), (4, 1)\}.$$

Then R is not symmetric, since $(1, 2) \in R$, but $(2, 1) \notin R$. Also, R is not asymmetric, since $(2, 2) \in R$. Finally, R is antisymmetric, since if $a \neq b$, either $(a, b) \notin R$ or $(b, a) \notin R$. ◆

Transitive Property of a Relation

Definition:

A relation *R* on a set *A* is called *transitive* if whenever (*a, b*) $\in$ *R* and (*b*, *c*) $\in$ *R* then (*a, c*) $\in$ *R*, for *a, b*, *c* $\in$ *A*.

Example : Which of the following relations are transitive?

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$

Not transitive because

- (3, 4) & (4, 1) $\in R_1$ but (3, 1) $\notin R_1$

$R_2 = \{(1, 1), (1, 2), (2, 1)\}$

Not transitive because

\- $(2, 1)$ & $(1, 2) \in R_2$ but $(2, 2) \notin R_2$

$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$

Not transitive

\- $(4, 1)$ & $(1, 2) \in R_3$ but $(4, 2) \notin R_3$

$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$

Transitive

$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$

Transitive

Examples 10

Example 10 Let $A = \{1, 2, 3, 4\}$ and let

$$R = \{(1, 2), (1, 3), (4, 2)\}.$$

Is R transitive?

Solution

Since there are no elements a, b, and c in A such that $a \; R \; b$ and $b \; R \; c$, but $a \not{R} \; c$, we conclude that R is transitive.

4.5 Equivalence Relations

Equivalence Relations

- **Equivalence relations** are used to relate objects that are similar in some way.
- **Definition:** A relation on a set A is called an equivalence relation if it is reflexive, symmetric, and transitive.
- Two elements that are related by an equivalence relation R are called **equivalent**.

Equivalence Relations

•Since R is **symmetric**, a is equivalent to b whenever b is equivalent to a.

•Since R is **reflexive**, every element is equivalent to itself.

•Since R is **transitive**, if a and b are equivalent and b and c are equivalent, then a and c are equivalent.

•Obviously, these three properties are necessary for a reasonable definition of equivalence.

Section 4.5: Examples 2

Example 2 Let $A = \{1, 2, 3, 4\}$ and let

$$R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 3), (3, 3), (4, 4)\}.$$

It is easy to verify that R is an equivalence relation.

4.7 Operations on Relations

Combining Relations

- Relations are sets, and therefore, we can apply the usual **set operations** to them.

- If we have two relations R_1 and R_2, and both of them are from a set A to a set B, then we can combine them to $R_1 \cup R_2$, $R_1 \cap R_2$, or $R_1 - R_2$.

- In each case, the result will be **another relation from A to B**.

Example

Let $R_1 = \{(1, 1), (2, 2), (3, 3)\}$ and

$R_2 = \{(1, 1), (1, 2), (1, 3), (1,4)\}$

then :

$$R_1 \cup R_2 = \{(1, 1), (1, 2), (1, 3), (1,4), (2, 2), (3, 3)\}$$

$$R_1 \cap R_2 = \{(1, 1)\}$$

$$R_1 \setminus R_2 = \{(2, 2), (3, 3)\}$$

$$R_2 \setminus R_1 = \{(1, 2), (1, 3), (1,4)\}$$

Example 1

Example 1 Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c\}$. Let

$$R = \{(1, a), (1, b), (2, b), (2, c), (3, b), (4, a)\}$$

and

$$S = \{(1, b), (2, c), (3, b), (4, b)\}.$$

Compute **(a)** $\overline{R}$; **(b)** $R \cap S$; **(c)** $R \cup S$; and **(d)** R^{-1}.

Solution

(a) We first find

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c), (3, a), (3, b), (3, c), (4, a), (4, b), (4, c)\}.$$

Then the complement of R in $A \times B$ is

$$\overline{R} = \{(1, c), (2, a), (3, a), (3, c), (4, b), (4, c)\}.$$

(b) We have $R \cap S = \{(1, b), (3, b), (2, c)\}$.

(c) We have

$$R \cup S = \{(1, a), (1, b), (2, b), (2, c), (3, b), (4, a), (4, b)\}.$$

(d) Since $(x, y) \in R^{-1}$ if and only if $(y, x) \in R$, we have

$$R^{-1} = \{(a, 1), (b, 1), (b, 2), (c, 2), (b, 3), (a, 4)\}.$$

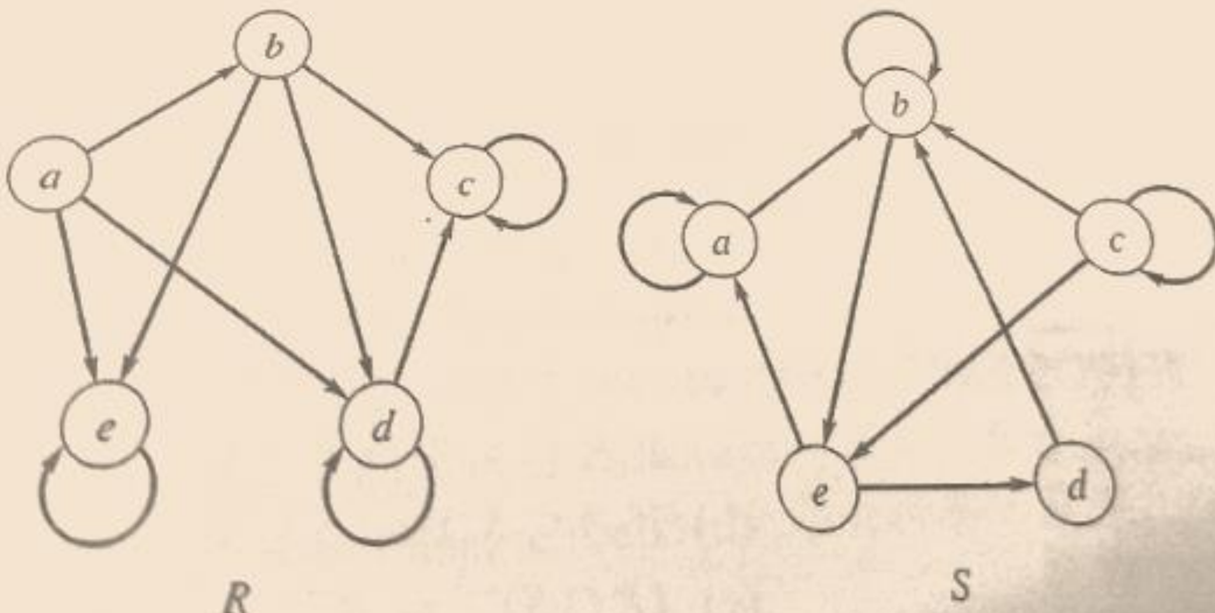
Example 3

Example 3 Let $A = \{a, b, c, d, e\}$ and let R and S be two relations on A whose corresponding digraphs are shown in Figure 4.38. Then the reader can verify the following facts:

$$\overline{R} = \{(a,a), (b,b), (a,c), (b,a), (c,b), (c,d), (c,e), (c,a), (d,b), (d,a), (d,e), (e,b), (e,a), (e,d), (e,c)\}$$

$$R^{-1} = \{(b,a), (e,b), (c,c), (c,d), (d,d), (d,b), (c,b), (d,a), (e,e), (e,a)\}$$

$$R \cap S = \{(a,b), (b,e), (c,c)\}.$$



Example 4

Example 4 Let $A = \{1, 2, 3\}$ and let R and S be relations on A. Suppose that the matrices of R and S are

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Then we can verify that

$$\mathbf{M}_{\overline{R}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \qquad \mathbf{M}_{R^{-1}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix},$$

$$\mathbf{M}_{R \cap S} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{M}_{R \cup S} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

◆

Example 5

Example 5 Let $A = \{1, 2, 3\}$ and consider the two reflexive relations

$$R = \{(1, 1), (1, 2), (1, 3), (2, 2), (3, 3)\}$$

and

$$S = \{(1, 1), (1, 2), (2, 2), (3, 2), (3, 3)\}.$$

Then

(a) $R^{-1} = \{(1, 1), (2, 1), (3, 1), (2, 2), (3, 3)\}$; R and R^{-1} are both reflexive.
(b) $\overline{R} = \{(2, 1), (2, 3), (3, 1), (3, 2)\}$ is irreflexive while R is reflexive.
(c) $R \cap S = \{(1, 1), (1, 2), (2, 2), (3, 3)\}$ and $R \cup S = \{(1, 1), (1, 2), (1, 3), (2, 2), (3, 2), (3, 3)\}$ are both reflexive. ◆

Example 6

Example 6 Let $A = \{1, 2, 3\}$ and consider the symmetric relations

$$R = \{(1, 1), (1, 2), (2, 1), (1, 3), (3, 1)\}$$

and

$$S = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}.$$

Then

(a) $R^{-1} = \{(1, 1), (2, 1), (1, 2), (3, 1), (1, 3)\}$ and $\overline{R} = \{(2, 2), (2, 3), (3, 2), (3, 3)\}$; R^{-1} and $\overline{R}$ are symmetric.

(b) $R \cap S = \{(1, 1), (1, 2), (2, 1)\}$ and $R \cup S = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1), (3, 3)\}$, which are both symmetric. ◆

Example 10

Example 10 Let $A = \{1, 2, 3, 4\}$, $R = \{(1, 2), (1, 1), (1, 3), (2, 4), (3, 2)\}$, and $S = \{(1, 4), (1, 3), (2, 3), (3, 1), (4, 1)\}$. Since $(1, 2) \in R$ and $(2, 3) \in S$, we must have $(1, 3) \in S \circ R$. Similarly, since $(1, 1) \in R$ and $(1, 4) \in S$, we see that $(1, 4) \in S \circ R$. Proceeding in this way, we find that $S \circ R = \{(1, 4), (1, 3), (1, 1), (2, 1), (3, 3)\}$. ◆

Example 12

Example 12 Let us redo Example 10 using matrices. We see that

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_S = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Then

$$\mathbf{M}_R \odot \mathbf{M}_S = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so

$$S \circ R = \{(1,1), (1,3), (1,4), (2,1), (3,3)\}$$

as we found before. In cases where the number of pairs in R and S is large, the matrix method is much more reliable. ◆

Example 13

Example 13 Let $A = \{a, b\}$, $R = \{(a, a), (b, a), (b, b)\}$, and $S = \{(a, b), (b, a), (b, b)\}$. Then $S \circ R = \{(a, b), (b, a), (b, b)\}$, while $R \circ S = \{(a, a), (a, b), (b, a), (b, b)\}$. ♦