4. Relations and Digraphs

Product Sets

- An ordered pair (a,b) is a listing of the objects a and b in a prescribed order.
- If A and B are two nonempty sets, the **product set** or **Cartesian product** $A \times B$ is the set of all ordered pairs (a,b) with $a \in A$, $b \in B$.

Theorem 1. For any two finite, nonempty sets A and B, $|A \times B| = |A||B|$

- **Cartesian product** of the nonempty sets $A_1, A_2, \ldots, A_m$ is the set of all ordered m-tuples $(a_1, a_2, \ldots, a_m)$ where $a_i \in A_i$, $i=1,2, \ldots, m$.

  $A_1 \times A_2 \times \ldots \times A_m = \{(a_1, a_2, \ldots, a_m) \mid a_i \in A_i, i=1,2, \ldots, m\}$

Relations

- Let A and B be nonempty sets, a **relation** *R* **from** A **to** B is a subset of $A \times B$. If $(a,b) \in R$, then *a* **is related to** *b* by *R* and *aRb*.
- If $R \subseteq A \times A$, *R* is **a relation on A**.
- The **domain** of R, Dom(*R*), is the set of elements in A that are related to some elements in B.
- The **range** of R, Ran(*R*), is the set of elements in B that are related to some elements in A.
- *R(x)* is defined as the ***R*-relative set of *x***, where $x \in A$, $R(x)=\{y \in B \mid xRy\}$
- $R(A_1)$ is defined as the ***R*-relative set of A_1**, where $A_1 \subseteq A$, $R(A_1)=\{y \in B \mid xRy$ for some x in $A_1\}$

Relations

Theorem 1. Let R be a relation from A to B, and let A_1 and A_2 be subsets of A. Then

(a) If $A_1 \subseteq A_2$, then $R(A_1) \subseteq R(A_2)$.

(b) $R(A_1 \cup A_2) = R(A_1) \cup R(A_2)$.

(c) $R(A_1 \cap A_2) \subseteq R(A_1) \cap R(A_2)$.

Theorem 2. Let R and S be relations form A to B. If $R(a)=S(a)$ for all *a* in A, then R=S.

The Matrix of a Relation

If A and B are finites sets containing m and n elements, respectively, and R is a relation from A to B, represent R by the $m \times n$ matrix $M_R=[m_{ij}]$, where $m_{ij}=1$ if $(a_i,b_j) \in R$; $m_{ij}=0$ if $(a_i,b_j) \notin R$.

M_R is called the **matrix of *R*.**

- Conversely, given sets A and B with $|A|=m$ and $|B|=n$, an $m \times n$ matrix whose entries are zeros and ones determines a relation: $(a_i,b_j) \in R$ if and only if $m_{ij}=1$.

The Digraph of a Relation

- Draw circles called **vertices** for elements of A, and draw arrows called **edges** from vertex a_i to vertex a_j if and only if $a_i R a_j$.
- The pictorial representation of *R* is called a **directed graph** or **digraph** of *R*.
- A collection of vertices and edges in a digraph determines a relation
- If *R* is a relation on A and $a \in A$, then the **in-degree of *a*** is the number of $b \in A$ such that $(b,a) \in R$; the **out-degree of *a*** is the number of $b \in A$ such that $(a,b) \in R$, the out-degree of a is $|R(a)|$
- The sum of all in-degrees in a digraph equals the sum of all out-degrees.
- If *R* is a relation on A, and B is a subset of A, the **restriction of *R* to B** is $R \cap (B \times B)$.

4.1Product sets and partitions

Relations on a Set

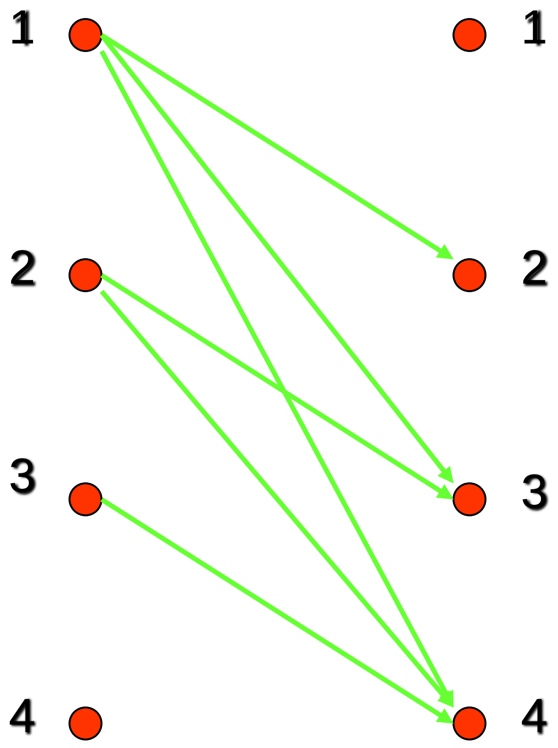
- **Definition:** A relation on the set A is a relation from A to A.

- In other words, a relation on the set A is a subset of $A \times A$.

- **Example:** Let A = {1, 2, 3, 4}. Which ordered pairs are in the relation $R = \{(a, b) \mid a < b\}$?

Relations on a Set

- **Solution:** R = {(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)}



R	1	2	3	4
1		X	X	X
2			X	X
3				X
4				

Example 1 Let

$$A = \{1, 2, 3\} \quad \text{and} \quad B = \{r, s\};$$

then

$$A \times B = \{(1, r), (1, s), (2, r), (2, s), (3, r), (3, s)\}.$$

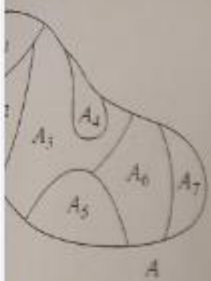
Observe that the elements of $A \times B$ can be arranged in a convenient tabular array as shown in Figure 4.1.

Example 2 If A and B are as in Example 1, then

$$B \times A = \{(r, 1), (s, 1), (r, 2), (s, 2), (r, 3), (s, 3)\}.$$

Partitions

- A **partition** or **quotient set** of a nonempty set *A* is a collection $\mathcal{P}$ of nonempty subsets of *A* such that
  - Each element of *A* belongs to one of the sets in $\mathcal{P}$.
  - If A_1 and A_2 are distinct elements of $\mathcal{P}$, then $A_1 \cap A_2 = \phi$.
- The sets in $\mathcal{P}$ are called the **blocks** or **cells** of the partition
- The members of a partition of a set A are subsets of A
- A partition is a subset of P(A), the power set of A
- Partitions can be considered as particular kinds of subsets of P(A)



4.2

Partitions

A **partition** or **quotient set** of a nonempty set A is a collection $\mathcal{P}$ of nonempty subsets of A such that

1. Each element of A belongs to one of the sets in $\mathcal{P}$.
2. If A_1 and A_2 are distinct elements of $\mathcal{P}$, then $A_1 \cap A_2 = \varnothing$.

The sets in $\mathcal{P}$ are called the **blocks** or **cells** of the partition. Figure 4.2 shows a partition $\mathcal{P} = \{A_1, A_2, A_3, A_4, A_5, A_6, A_7\}$ of A into seven blocks.

Example 6 Let $A = \{a, b, c, d, e, f, g, h\}$. Consider the following subsets of A:

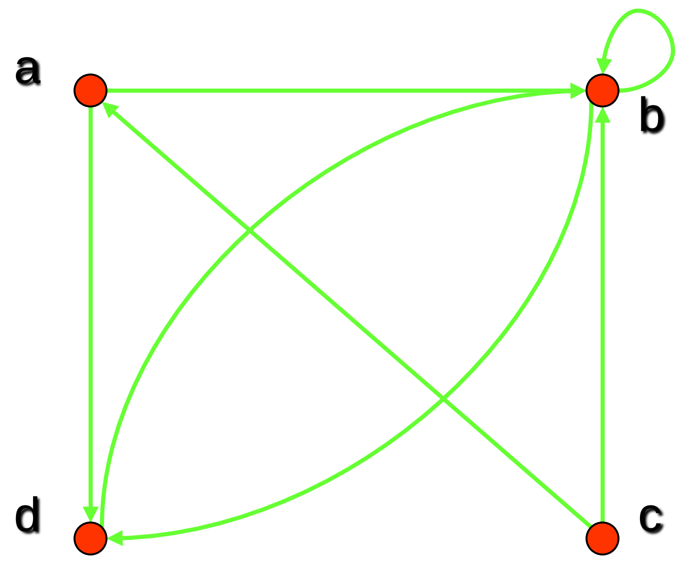
$$A_1 = \{a, b, c, d\}, \quad A_2 = \{a, c, e, f, g, h\}, \quad A_3 = \{a, c, e, g\},$$
$$A_4 = \{b, d\}, \quad A_5 = \{f, h\}.$$

Then $\{A_1, A_2\}$ is not a partition since $A_1 \cap A_2 \neq \varnothing$. Also, $\{A_1, A_5\}$ is not a partition since $e \notin A_1$ and $e \notin A_5$. The collection $\mathcal{P} = \{A_3, A_4, A_5\}$ is a partition of A.

4.2 Relations and diagraphs

Representing Relations Using Digraphs

- **Example:** Display the digraph with V = {a, b, c, d},
E = {(a, b), (a, d), (b, b), (b, d), (c, a), (c, b), (d, b)}.



An edge of the form (b, b) is called a **loop.**

... examples. ... considered a relation. We

Example 1 Let $A = \{1, 2, 3\}$ and $B = \{r, s\}$. Then $R = \{(1, r), (2, s), (3, r)\}$ is a relation from A to B. ◆

Example 2 Let A and B be sets of real numbers. We define the following relation R (equals) from A to B:

$$a \; R \; b \quad \text{if and only if} \quad a = b.$$ ◆

Example 3 Let $A = \{1, 2, 3, 4, 5\}$. Define the following relation R (less than) on A:

$$a \; R \; b \quad \text{if and only if} \quad a < b.$$

Then

$$R = \{(1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\}.$$ ◆

Example 4 Let $A = \mathbb{Z}^+$, the set of all positive integers. Define the following relation R on A:

$$a \; R \; b \quad \text{if and only if} \quad a \text{ divides } b.$$

Then $4 \; R \; 12$, but $5 \not R \; 7$. ◆

... following relation

Example 10 If R is the relation defined in Example 1, then $\text{Dom}(R) = A$ and $\text{Ran}(R) = B$. ◆

Example 11 If R is the relation given in Example 3, then $\text{Dom}(R) = \{1, 2, 3, 4\}$ and $\text{Ran}(R) = \{2, 3, 4, 5\}$. ◆

Example 18 Consider the matrix

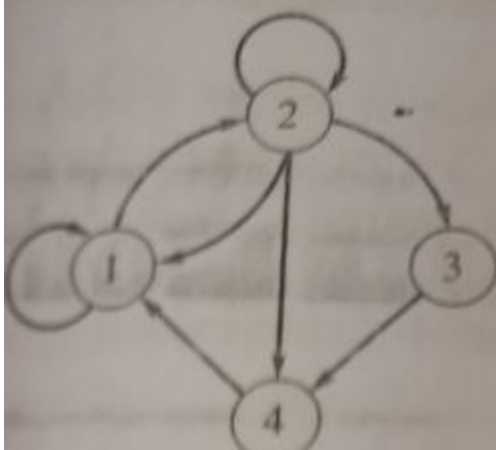
$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

Since $\mathbf{M}$ is 3×4, we let

$$A = \{a_1, a_2, a_3\} \quad \text{and} \quad B = \{b_1, b_2, b_3, b_4\}.$$

Then $(a_i, b_j) \in R$ if and only if $m_{ij} = 1$. Thus

$$R = \{(a_1, b_1), (a_1, b_4), (a_2, b_2), (a_2, b_3), (a_3, b_1), (a_3, b_3)\}.$$



igure 4.4

Example 19 Let

$$A = \{1, 2, 3, 4\}$$
$$R = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (2, 4), (3, 4), (4, 1)\}.$$

Then the digraph of R is as shown in Figure 4.4. ◆

A collection of vertices with edges between some of the vertices determines a relation in a natural manner.

degree 2, while

Example 22 Let $A = \{a, b, c, d\}$, and let R be the relation on A that has the matrix

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Construct the digraph of R, and list in-degrees and out-degrees of all vertices.

Solution

The digraph of R is shown in Figure 4.6. The following table gives the in-degrees and out-degrees of all vertices. Note that the sum of all in-degrees must equal the sum of all out-degrees.

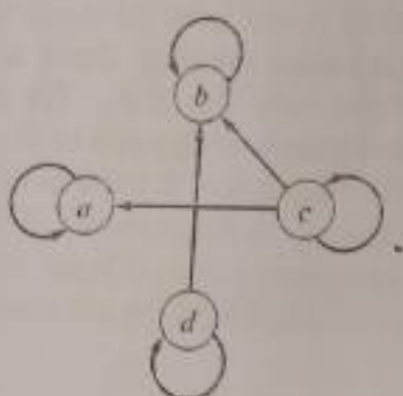


Figure 4.6

	a	b	c	d
In-degree	2	3	1	1
Out-degree	1	1	3	2

Example 23 Let $A = \{1, 4, 5\}$, and let R be given by the digraph shown in Figure 4.7. Find $\mathbf{M}_R$ and R.

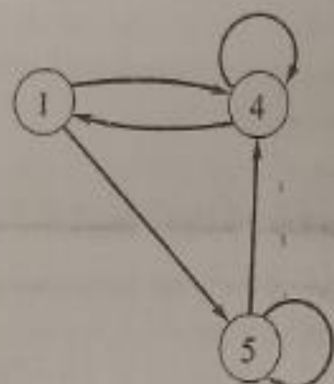


Figure 4.7

Solution

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad R = \{(1,4), (1,5), (4,1), (4,4), (5,4), (5,5)\}$$ ◆

If R is a relation on a set A, and B is a subset of A, the **restriction of R to B** is $R \cap (B \times B)$.

4.3 Paths in Relations and diagraphs

Paths in Relations and Digraphs

- A **path of length *n*** in R from a to b is a finite sequence π: $a, x_1, x_2, \ldots, x_{n-1}, b$ such that aRx_1, $x_1Rx_2, \ldots, x_{n-1}Rb$ where x_i are elements of A
- A path that begins and ends at the same vertex is called a **cycle**
- the **paths of length 1** can be identified with the ordered pairs (x,y) that belong to R
- $\boldsymbol{xR^ny}$ means that there is a path of length n from x to y in R; $\boldsymbol{R^n(x)}$ consists of all vertices that can be reached from x by some path in R of length n
- $\boldsymbol{xR^\infty y}$ means that there is some path from x to y in R, the length will depend on x and y; R^∞ is sometimes called the **connectivity relation** for R
- $\boldsymbol{R^\infty(x)}$ consists of all vertices that can be reached from x by some path in R

Paths in Relations and Digraphs

- If $|R|$ is large, M_R can be used to compute R^∞ and R^2 efficiently

Theorem1 If R is a relation on $A=\{a_1,a_2,\ldots,a_m\}$, then

$M_{R^2} = M_R \odot M_R$

Theorem2 For $n\geq 2$, and R a relation on a finite set A, we have

$M_{R^n}=M_R\odot M_R\odot\ldots\odot M_R$ (n factors)

- The **reachability** relation R^* of a relation R on a set A that has n elements is defined as follows: xR^*y means that $x=y$ or $xR^\infty y$
- Let π_1: $a,x_1,x_2,\ldots,x_{n-1},b$ be a path in a relation R of length n from a to b, and let π_2: $b,y_1,y_2,\ldots,y_{m-1},c$ be a path in R of length m from b to c, then the **composition of** π_1 **and** π_2 is the path of length $n+m$ from a to c, which is denoted by

  $\pi_2 \circ \pi_1$

4.3 Paths in Relations and Digraphs

Suppose that R is a relation on a set A. A **path of length n** in R from a to b is a finite sequence $\pi : a, x_1, x_2, \ldots, x_{n-1}, b$, beginning with a and ending with b, such that

$$a\ R\ x_1,\ x_1\ R\ x_2,\ \ldots,\ x_{n-1}\ R\ b.$$

Note that a path of length n involves $n+1$ elements of A, although they are not necessarily distinct.

Example 5 Let $A = \{a, b, c, d, e\}$ and

$$R = \{(a, a), (a, b), (b, c), (c, e), (c, d), (d, e)\}.$$

Compute (**a**) R^2; (**b**) R^∞.

Solution

(**a**) The digraph of R is shown in Figure 4.14.

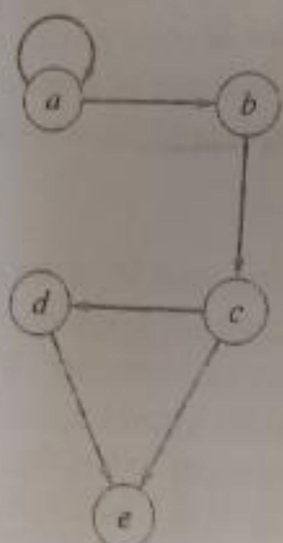


Figure 4.14

$$\begin{array}{llllll}
a\,R^2\,a & \text{since} & a\,R\,a & \text{and} & a\,R\,a. \\
a\,R^2\,b & \text{since} & a\,R\,a & \text{and} & a\,R\,b. \\
a\,R^2\,c & \text{since} & a\,R\,b & \text{and} & b\,R\,c. \\
b\,R^2\,e & \text{since} & b\,R\,c & \text{and} & c\,R\,e. \\
b\,R^2\,d & \text{since} & b\,R\,c & \text{and} & c\,R\,d. \\
c\,R^2\,e & \text{since} & c\,R\,d & \text{and} & d\,R\,e.
\end{array}$$

Hence

$$R^2 = \{(a, a), (a, b), (a, c), (b, e), (b, d), (c, e)\}.$$

(**b**) To compute R^∞, we need all ordered pairs of vertices for whic[h] a path of any length from the first vertex to the second. From Fi[g] we see that

$$R^\infty = \{(a,a), (a,b), (a,c), (a,d), (a,e), (b,c), (b,d), (b,e), (c,d), (c,e), (d,e)\}.$$

For example, $(a,d) \in R^\infty$, since there is a path of length 3 from a to d: a, b, c, d. Similarly, $(a,e) \in R^\infty$, since there is a path of length 3 from a to e: a, b, c, e as well as a path of length 4 from a to e: a, b, c, d, e.

If $|R|$ is large, it can be tedious and perhaps difficult to compute R^∞, or even R^2, from the set representation of R. However, $\mathbf{M}_R$ can be used to accomplish these tasks more efficiently.

Let R be a relation on a finite set $A = \{a_1, a_2, \ldots, a_n\}$, and let $\mathbf{M}_R$ be the $n \times n$ matrix representing R. We will show how the matrix $\mathbf{M}_{R^2}$, of R^2, can be computed from $\mathbf{M}_R$.

reminds us that this is not the usual matrix product).

Example 6 Let A and R be as in Example 5. Then

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

From the preceding discussion, we see that

$$\mathbf{M}_{R^2} = \mathbf{M}_R \odot \mathbf{M}_R = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \odot \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Computing $\mathbf{M}_{R^2}$ directly from R^2, we obtain the same result.

$$\mathbf{M}_{R^2} = \mathbf{M}_R \odot \mathbf{M}_R = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \odot \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Computing $\mathbf{M}_{R^2}$ directly from R^2, we obtain the same result. ◆

We can see from Examples 5 and 6 that it is often easier to compute R^2 by computing $\mathbf{M}_R \odot \mathbf{M}_R$ instead of searching the digraph of R for all vertices that can be joined by a path of length 2. Similarly, we can show that $\mathbf{M}_{R^3} = \mathbf{M}_R \odot (\mathbf{M}_R \odot \mathbf{M}_R) = (\mathbf{M}_R)^3_{\odot}$. In fact, we now show by induction that these two results can be generalized.

First Exam

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<tr><td colspan="4">طلاب شعبة رقم (2) الدكتورة نجاح الشنابلة قاعات رقم (101) و (102) مبنى الكلية</td></tr>
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