Permutations and Combinations

Permutations vs. Combinations

- Both are ways to count the possibilities
- The difference between them is whether order matters or not
- Consider a poker hand:
  - A♦, 5♥, 7♣, 10♠, K♠
- Is that the same hand as:
  - K♠, 10♠, 7♣, 5♥, A♦
- Does the order the cards are handed out matter?
  - If yes, then we are dealing with permutations
  - If no, then we are dealing with combinations

3.1

Permutations

Permutations

- Number of poker hands (5 cards):
  - *P*(52,5) = 52*51*50*49*48 = 311,875,200
- Number of (initial) blackjack hands (2 cards):
  - *P*(52,2) = 52*51 = 2,652
- *r*-permutation notation: *P*(*n*,*r*)
  - The poker hand is one of P(52,5) permutations

$$P(n,r) = n(n-1)(n-2)\ldots(n-r+1)$$

$$= \frac{n!}{(n-r)!}$$

$$= \prod_{i=n-r+1}^{n} i$$

Permutations with indistinguishable objects

How many distinct permutations are there of the letters in the word APALACHICOLA?

$$\frac{12!}{4!2!2!}$$

How many if the two Ls must appear together?

$$\frac{11!}{4!2!}$$

How many if the first letter must be an A?

$$\frac{11!}{3!2!2!}$$

Permutations

- A permutation is an ordered arrangement of the elements of some set *S*
  - Let *S* = {a, b, c}
  - c, b, a is a permutation of S
  - b, c, a is a *different* permutation of S
- An *r*-permutation is an ordered arrangement of *r* elements of the set
  - A♦, 5♥, 7♣, 10♠, K♠ is a 5-permutation of the set of cards
- The notation for the number of *r*-permutations: $P(n,r)$
  - The poker hand is one of P(52,5) permutations

3.1 Examples

- Example: How many "words" of three distinct letters can be formed from the letters of the word MAST

$$4P3 = 4!/(4-3)! = 4!/1! = 24$$

3.1 Examples

- How many distinguishable permutations of the letters in the word BANANA are there ?
  (A) 720
  (B) 120
  (C) 60
  (D) 360

- **Explanation:** In BANANA we have six letters in total but here we have some duplicate letters too so we have to deal with it and have to remove those duplicate case.
  B – 1
  A – 3
  N – 2
  So total no of words possible is factorial(6) ie 6! but we must remove duplicate words:
  ie- (6!/(2!*3!))
  which gives 60
  So 60 distinguishable permutation of the letters in BANANA.
  So, option (A) is correct.

Theory

- The number of distinguishable permutations that can be formed from a collection of n objects where the first object appears k1 times, and the second object appears k2 times, and so on, is:

$$\frac{n!}{k_1!k_2!k_3!\cdots k_r!}$$

3.1 Examples

- How many distinguishable permutations of the letters in the word MISSISSIPPI are there?

11!/(1!4!4!2!)= 34.650

Here are the frequencies of the letters. M=1, I=4, S=4, P=2 for a total of 11 letters. Be sure you put parentheses around the denominator so that you end up dividing by each of the factorials.
11! / (1! * 4! * 4! * 2!) = 11! / (1 * 24 * 24 * 2) = 34,650.

3.2

Combinations

Combinations

- What if order *doesn't* matter?
- In poker, the following two hands are equivalent:
  - A♦, 5♥, 7♣, 10♠, K♠
  - K♠, 10♠, 7♣, 5♥, A♦

- The number of *r*-combinations of a set with *n* elements, where *n* is non-negative and $0 \le r \le n$ is:

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

Combinations example

- How many different poker hands are there (5 cards)?

$$C(52,5) = \frac{52!}{5!(52-5)!} = \frac{52!}{5!47!} = \frac{52*51*50*49*48*47!}{5*4*3*2*1*47!} = 2,598,960$$

- How many different (initial) blackjack hands are there?

$$C(52,2) = \frac{52!}{2!(52-2)!} = \frac{52!}{2!50!} = \frac{52*51}{2*1} = 1,326$$

Combinations

- The number of ways of choosing r elements from S (order does not matter).
- S={1,2,3}
- e.g., 1 2 , 1 3, 2 3

$$C(n,r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

- The number of r-combinations $C(n,r)$ of a set with $n=|S|$ elements is = P(n,r) / r!

Note: we have $C(n,r) = C(n, n-r)$

"n choose r". Also called a "binomial coefficient".

Combination Example

- How many distinct 7-card hands can be drawn from a standard 52-card deck?
  - The order of cards in a hand doesn't matter.

- Answer $C(52,7) = P(52,7) / P(7,7)$

- $= 52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot 47 \cdot 46 / 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
- $= 52.17.10.7.2.47.23$
- $= 133,784,560$

- Combination Example:

Pick a team of 3 people from a group of 10. How many outcomes are possible?

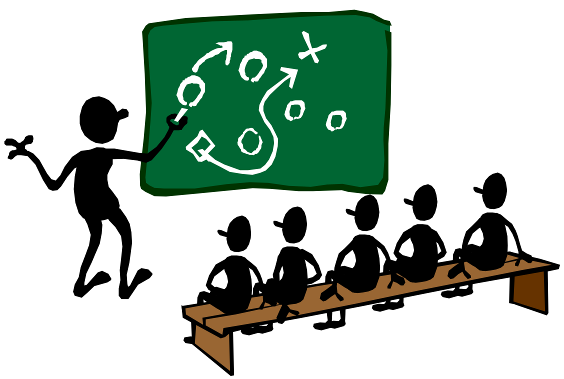
- It is a combination since the order that the 3 people are selected does not matter.

$_{10}C_3$ $\qquad$ $\frac{n!}{(n-r)!\,r!}$

$\frac{10!}{(10-3)!3!}$ we know the (10 – 3)! Cancels 7 and

below so $\frac{10 \bullet 9 \bullet 8}{3 \bullet 2 \bullet 1}$

$\frac{720}{6}$ = 120 outcomes



3.2 Examples

- Compute the number of distinct five-card hands that can be dealt from a deck of 52 cards.

$$C(n,r)=\frac{n!}{r!(n-r)!}$$

${}_{52}C_5$ = 52!/5!47!

${}_{52}C_5$ = 2,598,960

Horse races

- How many ways are there for 4 horses to finish if ties are allowed?
  - Note that order does matter!
- Solution by cases
  - No ties
    - The number of permutations is P(4,4) = 4! = 24
  - Two horses tie
    - There are *C*(4,2) = 6 ways to choose the two horses that tie
    - There are *P*(3,3) = 6 ways for the "groups" to finish
      - A "group" is either a single horse or the two tying horses
    - By the product rule, there are 6*6 = 36 possibilities for this case
  - Two groups of two horses tie
    - There are *C*(4,2) = 6 ways to choose the two winning horses
    - The other two horses tie for second place
  - Three horses tie with each other
    - There are *C*(4,3) = 4 ways to choose the two horses that tie
    - There are *P*(2,2) = 2 ways for the "groups" to finish
    - By the product rule, there are 4*2 = 8 possibilities for this case
  - All four horses tie
    - There is only one combination for this
  - By the sum rule, the total is 24+36+6+8+1 = 75