2.2

Conditional Statements

Special Characteristics of Conditional Statements for a Truth Table

Teacher:

“If you participate in class, then you will get extra points."

When the antecedent is *true* and the consequent is *true*, $p \rightarrow q$ is true.

If you participate in class (true) and you get extra points (true) then,

The teacher's statement is true.

When the antecedent is *true* and the consequent is *false*, $p \rightarrow q$ is false.

If you participate in class (true) and you do not get extra points (false), then,

The teacher’s statement is false.

Special Characteristics of Conditional Statements for a Truth Table

"If you participate in class, then you will get extra points."

If the antecedent is *false*, then $\boldsymbol{p} \rightarrow \boldsymbol{q}$ is automatically *true*.

If you do not participate in class (false), the truth of the teacher's statement cannot be judged.

The teacher did not state what would happen if you did NOT participate in class. Therefore, the statement has to be "true".

If you do not participate in class (false), then you get extra points.

If you do not participate in class (false), then you do not get extra points.

The teacher's statement is true in both cases.

The Conditional

- If p, then q
- Symbols: $p \rightarrow q$
- p is the antecedent, q is the consequent

Truth Table for The Conditional

If *p*, then *q*

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

A **tautology** is a statement that is always true, no matter what the truth values of the components are.

Examples:

Decide whether each statement is True or False

(T represents a true statement, F a false statement).

$$T \rightarrow (4 < 2)$$
$$T \rightarrow F$$
$$F$$

$$(8 = 1) \rightarrow F$$
$$F \rightarrow F$$
$$T$$

$$F \rightarrow (3 \neq 9)$$
$$F \rightarrow T$$
$$T$$

Converse, Inverse, and Contrapositive

Conditional Statement	$p \rightarrow q$	If *p*, then *q*
Converse	$q \rightarrow p$	If *q*, then *p*
Inverse	$\sim p \rightarrow \sim q$	If not *p*, then not *q*
Contrapositive	$\sim q \rightarrow \sim p$	If not *q*, then not *p*

Truth Table for Conditional

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Useful results for the Conditional

- **Equivalent to a disjunction**:

$$\boldsymbol{p \rightarrow q \equiv {\sim}p \vee q}$$

- **Negation**:

$$\boldsymbol{{\sim}(p \rightarrow q) \equiv p \wedge {\sim}q}$$

Related Conditional Statements

Direct statement	$p \rightarrow q$	If *p*, then *q*
Converse	$q \rightarrow p$	If *q*, then *p*
Inverse	$\sim p \rightarrow \sim q$	If not *p*, then not *q*
Contrapositive	$\sim q \rightarrow \sim p$	If not *q*, then not *p*

Equivalences

Direct statement and contrapositive are equivalent:

$$\boldsymbol{p \rightarrow q \equiv \sim q \rightarrow \sim p}$$

Converse and Inverse are equivalent:

$$\boldsymbol{q \rightarrow p \equiv \sim p \rightarrow \sim q}$$

Common wording for $p \to q$

If *p*, then *q*	*p* is sufficient for *q*
If *p*, *q*	*q* is necessary for *p*
p implies *q*	All *p*'s are *q*'s
p only if *q*	*q* if *p*

Examples

Example 1 Form the implication $p \Rightarrow q$ for each of the following.

(a) p: I am hungry. q: I will eat.

(b) p: It is snowing. q: $3 + 5 = 8$.

Solution

(a) If I am hungry, then I will eat.

(b) If it is snowing, then $3 + 5 = 8$.

Example 2 Give the converse and the contrapositive of the implication "If it is raining, then I get wet."

Solution

We have p: It is raining; and q: I get wet. The converse is $q \Rightarrow p$: If I get wet, then it is raining. The contrapositive is $\sim q \Rightarrow \sim p$: If I do not get wet, then it is not raining.

Example 3 Is the following equivalence a true statement? $3 > 2$ if and only if $0 < 3 - 2$.

Solution

Let p be the statement $3 > 2$ and let q be the statement $0 < 3 - 2$. Since both p and q are true, we conclude that $p \Leftrightarrow q$ is true. ◆

Example 4 Compute the truth table of the statement $(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$.

Solution

The following table is constructed using steps 1, 2, and 3 as given in Section 2.1. The numbers below the columns show the order in which they were constructed.

p	q	$p \Rightarrow q$	$\sim q$	$\sim p$	$\sim q \Rightarrow \sim p$	$(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$
T	T	T	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T
		(1)	(2)	(3)	(4)	(5)

◆

A statement that is true for all possible values of its propositional variables is called a **tautology**. A statement that is always false is called a **contradiction** or an **absurdity**, and a statement that can be either true or false, depending on the truth values of its propositional variables, is called a **contingency**.

2.4

Mathematical Induction

Mathematical induction is a form of mathematical proof.

Just because a rule, pattern, or formula seems to work for several values of n, you cannot simply decide that it is valid for all values of n without going through a legitimate proof.

The Principle of Mathematical Induction

Let P_n be a statement involving the positive integer n. If

1. P_1 is true, and
2. the truth of P_k implies the truth of P_{k+1}, for every positive integer k,

then P_n must be true for all integers n.

What is induction?

- A method of proof

- It does not generate answers: it only can prove them

- Three parts:
  - Base case(s): show it is true for one element
  - Inductive hypothesis: assume it is true for any given element
    - **Must be clearly labeled!!!**
  - Show that if it true for the next highest element

Induction example

- Show that the sum of the first *n* odd integers is n^2
  - Example: If $n = 5$, $1+3+5+7+9 = 25 = 5^2$
  - Formally, Show

$$\forall n \, P(n) \text{ where } P(n) = \sum_{i=1}^{n} 2i - 1 == n^2$$

- Base case: Show that P(1) is true

$$P(1) = \sum_{i=1}^{1} 2(i) - 1 == 1^2$$

$$= 1 == 1$$

Induction example, continued

- Inductive hypothesis: assume true for *k*
  - Thus, we assume that P(*k*) is true, or that

$$\sum_{i=1}^{k} 2i - 1 == k^2$$

  - Note: we don't yet know if this is true or not!

- Inductive step: show true for *k*+1
  - We want to show that:

$$\sum_{i=1}^{k+1} 2i - 1 == (k+1)^2$$

Induction example, continued

- Recall the inductive hypothesis:

$$\sum_{i=1}^{k} 2i - 1 == k^2$$

- Proof of inductive step:

$$\sum_{i=1}^{k+1} 2i - 1 == (k+1)^2$$

$$2(k+1) - 1 + \boxed{\sum_{i=1}^{k} 2i - 1} == k^2 + 2k + 1$$

$$2(k+1) - 1 + \boxed{k^2} == k^2 + 2k + 1$$

$$k^2 + 2k + 1 == k^2 + 2k + 1$$

What did we show

- Base case: P(1)
- If P(*k*) was true, then P(*k*+1) is true
  - i.e., P(*k*) → P(*k*+1)

- We know it's true for P(1)
- Because of P(*k*) → P(*k*+1), if it's true for P(1), then it's true for P(2)
- Because of P(*k*) → P(*k*+1), if it's true for P(2), then it's true for P(3)
- Because of P(*k*) → P(*k*+1), if it's true for P(3), then it's true for P(4)
- Because of P(*k*) → P(*k*+1), if it's true for P(4), then it's true for P(5)
- And onwards to infinity

- Thus, it is true for all possible values of *n*

- In other words, we showed that:

$$\left[P(1) \wedge \forall k \left(P(k) \rightarrow P(k+1)\right)\right] \rightarrow \forall n\, P(n)$$

Ex. Use mathematical induction to prove the following formula.

$$S_n = 1 + 3 + 5 + 7 + \cdots + (2n-1) = n^2$$

First, we must show that the formula works for n = 1.

1. For n = 1

$$S_1 = 1 = 1^2$$

The second part of mathematical induction has two steps. The first step is to assume that the formula is valid for some integer k. The second step is to use this assumption to prove that the formula is valid for the next integer, k + 1.

2. Assume $S_k = 1 + 3 + 5 + 7 + \cdots + (2k-1) = k^2$ is true, show that $S_{k+1} = (k + 1)^2$ is true.

$$S_{k+1} = 1 + 3 + 5 + 7 + \cdots + (2k - 1) + [2(k + 1) - 1]$$

$$= [1 + 3 + 5 + 7 + \cdots + (2k - 1)] + (2k + 2 - 1)$$

$$= S_k + (2k + 1)$$

$$= k^2 + 2k + 1$$

$$= (k + 1)^2$$

Ex. Use mathematical induction to prove the following formula.

$$S_n = 1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

1. Show n = 1 is true.

$$S_n = 1^2 = \frac{1(2)(3)}{6}$$

2. Assume that S_k is true.

$$S_k = 1^2 + 2^2 + 3^2 + 4^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

Show that $S_{k+1} = \frac{(k+1)(k+2)(2k+3)}{6}$ is true.

$$S_{k+1} = (1^2 + 2^2 + 3^2 + 4^2 + \cdots + k^2) + (k + 1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + (k + 1)^2$$

$$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$

Factor out a (k + 1)

$$= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6}$$

$$= \frac{(k+1)[2k^2 + 7k + 6]}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$$

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Sums of Powers of Integers

Ex.

$$\sum_{1}^{10} n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{10(10+1)(2(10)+1)}{6}$$

$= 385$

Examples

Example 1 Show, by mathematical induction, that for all $n \geq 1$,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Solution

Let P(n) be the predicate $1+2+3+\cdots+n = \dfrac{n(n+1)}{2}$. In this example, $n_0 = 1$.

Basis Step

We must first show that P(1) is true. P(1) is the statement

$$1 = \frac{1(1+1)}{2},$$

which is clearly true.

Induction Step

We must now show that for $k \geq 1$, if $P(k)$ is true, then $P(k+1)$ must also be true. We assume that for some fixed $k \geq 1$,

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}. \qquad (1)$$

We now wish to show the truth of $P(k+1)$:

$$1 + 2 + 3 + \cdots + (k+1) = \frac{(k+1)((k+1)+1)}{2}.$$

The left-hand side of $P(k+1)$ can be written as $1 + 2 + 3 + \cdots + k + (k+1)$ and we have

$$(1 + 2 + 3 + \cdots + k) + (k+1)$$

$$= \frac{k(k+1)}{2} + (k+1) \qquad \text{using (1) to replace } 1 + 2 + \cdots + k$$

$$= (k+1)\left[\frac{k}{2} + 1\right] \qquad \text{factoring}$$

$$= \frac{(k+1)(k+2)}{2}$$

$$= \frac{(k+1)((k+1)+1)}{2} \qquad \text{the right-hand side of } P(k+1)$$

Thus, we have shown the left-hand side of $P(k+1)$ equals the right-hand side of $P(k+1)$. By the principle of mathematical induction, it follows that $P(n)$ is true for all $n \geq 1$. ♦

Example 2 Let $A_1, A_2, A_3, \ldots, A_n$ be any n sets. We show by mathematical induction that

$$\overline{\left(\bigcup_{i=1}^{n} A_i\right)} = \bigcap_{i=1}^{n} \overline{A_i}.$$

(This is an extended version of one of De Morgan's laws.) Let P(n) be the predicate that the equality holds for any n sets. We prove by mathematical induction that for all $n \geq 1$, P(n) is true.

Basis Step

P(1) is the statement $\overline{A_1} = \overline{A_1}$, which is obviously true.

Induction Step

We use P(k) to show P($k+1$). The left-hand side of P($k+1$) is

$$\overline{\left(\bigcup_{i=1}^{k+1} A_i\right)} = \overline{A_1 \cup A_2 \cup \cdots \cup A_k \cup A_{k+1}}$$

$$= \overline{(A_1 \cup A_2 \cup \cdots \cup A_k) \cup A_{k+1}} \quad \text{associative property of } \cup$$

$$= \overline{(A_1 \cup A_2 \cup \cdots \cup A_k)} \cap \overline{A_{k+1}} \quad \text{by De Morgan's law for two sets}$$

$$= \left(\bigcap_{i=1}^{k} \overline{A_i}\right) \cap \overline{A_{k+1}} \quad \text{using P}(k)$$

$$= \bigcap_{i=1}^{k+1} \overline{A_i} \quad \text{right-hand side of P}(k+1)$$

Thus, the implication P(k) $\Rightarrow$ P($k+1$) is a tautology, and by the principle of mathematical induction P(n) is true for all $n \geq 1$. ◆