Geometric Sequences

Geometric Sequence– a sequence whose consecutive terms have a common *ratio*.

Geometric Sequence

A sequence is geometric if the ratios of consecutive terms are the same.

$$\frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3} = \ldots\ldots = r$$

The number *__r__* is the *__common ratio__*.

Ex. 1

Are these geometric?

2, 4, 8, 16, …, formula?, … Yes 2^n

12, 36, 108, 324, …, formula?, … **Yes $4(3)^n$**

$$-\frac{1}{3},\frac{1}{9},-\frac{1}{27},\frac{1}{61},\ldots,\ formula?,\ldots$$ No $(-1)^n/3$

1, 4, 9, 16, …, formula? , … No n^2

Finding the nth term of a Geometric Sequence

$$a_n = a_1 r^{n-1}$$

$$r = \frac{a_2}{a_1}$$

Ex. 2b

Write the first five terms of the geometric sequence whose first term is $a_1 = 9$ and $r = (1/3)$.

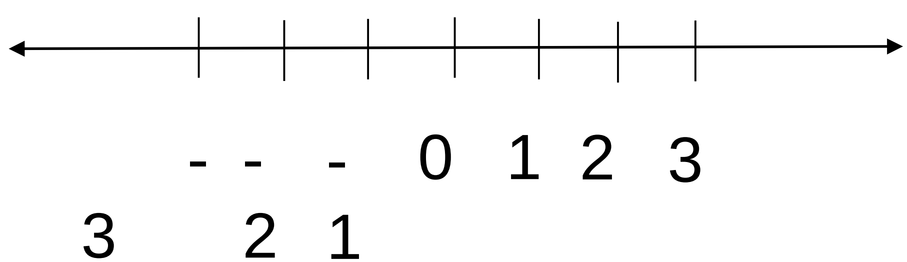
$$9, 3, 1, \frac{1}{3}, \frac{1}{9}$$

INTRODUCTION TO INTEGERS

- Integers are positive and negative numbers.

…, -6, -5, -4, -3, -2, -1, 0, +1, +2, +3, +4, +5, +6, …

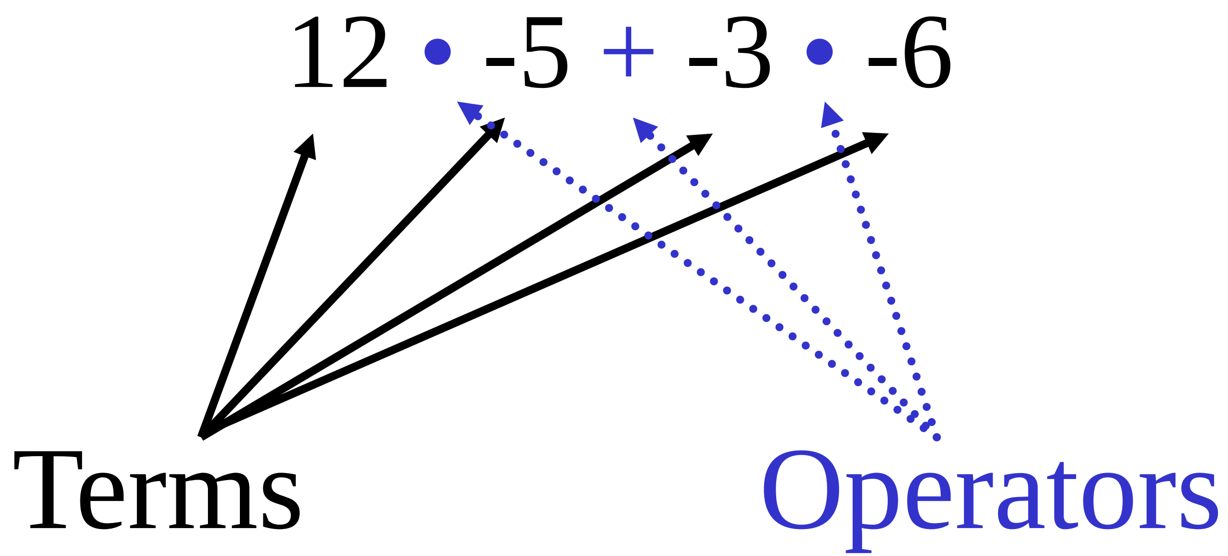
- Each negative number is paired with a positive number the same distance from 0 on a number line.



Integers

- Integers are the whole numbers and their opposites (no decimal values!)
- Example: -3 is an integer
- Example: 4 is an integer
- Example: 7.3 is not an integer

“Operators” & “Terms”…



Divisibility:
An integer a divides b (written “a|b”)
if and only if there exists an
Integer c such that c*a = b.

Primes:
A natural number $p \geq 2$ such that
among all the numbers 1,2…p
only 1 and p divide p.

(a mod n) means the remainder when a is divided by n.

$$a \bmod n = r$$

$$\Leftrightarrow$$

$$a = dn + r \text{ for some integer } d$$

Definition: Modular equivalence

$a \equiv b$ [mod n]

$\Leftrightarrow$ (a mod n) = (b mod n)

$\Leftrightarrow n \mid (a-b)$

$31 \equiv 81$ [mod 2]

$31 \equiv_2 81$

$31 \equiv 80$ [mod 7]

$31 \equiv_7 80$

Written as $a \equiv_n b$, and spoken "a and b are equivalent or congruent modulo n"

Greatest Common Divisor:
$GCD(x,y) =$
greatest $k \geq 1$ s.t. $k|x$ and $k|y$.

Least Common Multiple:
$LCM(x,y) =$
smallest $k \geq 1$ s.t. $x|k$ and $y|k$.

Fact:

$$GCD(x,y) \times LCM(x,y) = x \times y$$

You can use

$$MAX(a,b) + MIN(a,b) = a+b$$

applied appropriately to the factorizations of x and y to prove the above fact…

4) Find the GCF of 42 and 60.

$42 = 2 \cdot 3 \cdot 7$

$60 = 2 \cdot 2 \cdot 3 \cdot 5$

What prime factors do the numbers have in common?

Multiply those numbers.

The GCF is $2 \cdot 3 =$ **6**

6 is the largest number that can go into 42 and 60!

5) Find the GCF of $40a^2b$ and $48ab^4$.

$$40a^2b = 2 \bullet 2 \bullet 2 \bullet 5 \bullet a \bullet a \bullet b$$

$$48ab^4 = 2 \bullet 2 \bullet 2 \bullet 2 \bullet 3 \bullet a \bullet b \bullet b \bullet b \bullet b$$

What do they have in common?

Multiply the factors together.

GCF = **8ab**

What is the GCF of 48 and 64?

1. 2
2. 4
3. 8
4. 16

Example 7 Let $a = 540$ and $b = 504$. Factoring a and b into primes, we obtain

$$a = 540 = 2^2 \cdot 3^3 \cdot 5 \quad \text{and} \quad b = 504 = 2^3 \cdot 3^2 \cdot 7.$$

Thus all the prime numbers that are factors of either a or b are $p_1 = 2$, ... $p_3 = 5$, and $p_4 = 7$. Then $a = 2^2 \cdot 3^3 \cdot 5^1 \cdot 7^0$ and $b = 2^3 \cdot 3^2 \cdot 5^0 \cdot 7^1$ ... have

$$\begin{aligned} \text{GCD}(540, 504) &= 2^{\min(2,3)} \cdot 3^{\min(3,2)} \cdot 5^{\min(1,0)} \cdot 7^{\min(0,1)} \\ &= 2^2 \cdot 3^2 \cdot 5^0 \cdot 7^0 \\ &= 2^2 \cdot 3^2 \text{ or } 36. \end{aligned}$$

Also,

$$\begin{aligned} \text{LCM}(540, 504) &= 2^{\max(2,3)} \cdot 3^{\max(3,2)} \cdot 5^{\max(1,0)} \cdot 7^{\max(0,1)} \\ &= 2^3 \cdot 3^3 \cdot 5^1 \cdot 7^1 \text{ or } 7560. \end{aligned}$$

Then

$$\text{GCD}(540, 504) \cdot \text{LCM}(540, 504) = 36 \cdot 7560 = 272{,}160 = 540 \cdot 504.$$

As a verification, we can also compute GCD(540, 504) by the Euclidean algorithm and obtain the same result.

If n and m are integers and $n > 1$, Theorem 1 tells us we can write $m = qn + r$, $0 \leq r < n$. Sometimes the remainder r is more important than the quotient q.

Matrices

Introduction

Matrices - Introduction

Matrix algebra has at least two advantages:

•Reduces complicated systems of equations to simple expressions

•Adaptable to systematic method of mathematical treatment and well suited to computers

Definition:

A matrix is a set or group of numbers arranged in a square or rectangular array enclosed by two brackets

$$\begin{bmatrix} 1 & -1 \end{bmatrix} \qquad \begin{bmatrix} 4 & 2 \\ -3 & 0 \end{bmatrix} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Matrices - Introduction

Properties:

•A specified number of rows and a specified number of columns

•Two numbers (rows x columns) describe the dimensions or size of the matrix.

Examples:

3x3 matrix

2x4 matrix

1x2 matrix

$$\begin{bmatrix} 1 & 2 & 4 \\ 4 & -1 & 5 \\ 3 & 3 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 3 & -3 \\ 0 & 0 & 3 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & -1 \end{bmatrix}$$

Matrices - Introduction

A matrix is denoted by a bold capital letter and the elements within the matrix are denoted by lower case letters

e.g. matrix [**A**] with elements a_{ij}

$$\mathbf{A}_{mxn} = {}_{m}\mathbf{A}^{n} \quad \begin{bmatrix} a_{11} & a_{12}\ldots & a_{ij} & a_{in} \\ a_{21} & a_{22}\ldots & a_{ij} & a_{2n} \\ \mathrm{M} & \mathrm{M} & \mathrm{M} & \mathrm{M} \\ a_{m1} & a_{m2} & a_{ij} & a_{mn} \end{bmatrix}$$

i goes from 1 to m

j goes from 1 to n

Matrices - Introduction

TYPES OF MATRICES

1. **Column matrix or vector**:

The number of rows may be any integer but the number of columns is always 1

$$\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ -3 \end{bmatrix} \qquad \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$$

Matrices - Introduction

TYPES OF MATRICES

2. Row matrix or vector

Any number of columns but only one row

$$\begin{bmatrix} 1 & 1 & 6 \end{bmatrix} \qquad \begin{bmatrix} 0 & 3 & 5 & 2 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \Lambda & a_{1n} \end{bmatrix}$$

Matrices - Introduction

TYPES OF MATRICES

3. Rectangular matrix

Contains more than one element and number of rows is not equal to the number of columns

$$\begin{bmatrix} 1 & 1 \\ 3 & 7 \\ 7 & -7 \\ 7 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 0 & 3 & 3 & 0 \end{bmatrix}$$

$$m \neq n$$

Matrices - Introduction

TYPES OF MATRICES

4. Square matrix

The number of rows is equal to the number of columns

(a square matrix $\mathbf{A}_{m \times m}$ has an order of m)

$$\begin{bmatrix} 1 & 1 \\ 3 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 \\ 9 & 9 & 0 \\ 6 & 6 & 1 \end{bmatrix}$$

The principal or main diagonal of a square matrix is composed of all elements a_{ij} for which $i=j$

Matrices - Introduction

TYPES OF MATRICES

5. Diagonal matrix

A square matrix where all the elements are zero except those on the main diagonal

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

i.e. $a_{ij} = 0$ for all $i \neq j$

$a_{ij} \neq 0$ for some or all $i = j$

Matrices - Introduction

TYPES OF MATRICES

6. Unit or Identity matrix - I

A diagonal matrix with ones on the main diagonal

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} a_{ij} & 0 \\ 0 & a_{ij} \end{bmatrix}$$

i.e. $a_{ij} = 0$ for all $i \neq j$

$a_{ij} = 1$ for some or all $i = j$

Matrices - Introduction

TYPES OF MATRICES

7. Null (zero) matrix - 0

All elements in the matrix are zero

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$a_{ij} = 0$$ For all i,j

Matrices - Introduction

TYPES OF MATRICES

8. Triangular matrix

A square matrix whose elements above or below the main diagonal are all zero

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 8 & 9 \\ 0 & 1 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$

Matrices - Introduction

TYPES OF MATRICES

8a. Upper triangular matrix

A square matrix whose elements below the main diagonal are all zero

$$\begin{bmatrix} a_{ij} & a_{ij} & a_{ij} \\ 0 & a_{ij} & a_{ij} \\ 0 & 0 & a_{ij} \end{bmatrix} \quad \begin{bmatrix} 1 & 8 & 7 \\ 0 & 1 & 8 \\ 0 & 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 7 & 4 & 4 \\ 0 & 1 & 7 & 4 \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

i.e. $a_{ij} = 0$ for all $i > j$

Matrices - Introduction

TYPES OF MATRICES

8b. Lower triangular matrix

A square matrix whose elements above the main diagonal are all zero

$$\begin{bmatrix} a_{ij} & 0 & 0 \\ a_{ij} & a_{ij} & 0 \\ a_{ij} & a_{ij} & a_{ij} \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix}$$

i.e. $a_{ij} = 0$ for all $i < j$

Matrices – Introduction

TYPES OF MATRICES

9. Scalar matrix

A diagonal matrix whose main diagonal elements are equal to the same scalar

A scalar is defined as a single number or constant

$$\begin{bmatrix} a_{ij} & 0 & 0 \\ 0 & a_{ij} & 0 \\ 0 & 0 & a_{ij} \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

i.e. $a_{ij} = 0$ for all $i \neq j$

$a_{ij} = a$ for all $i = j$

Matrices

Matrix Operations

Matrices - Operations

EQUALITY OF MATRICES

Two matrices are said to be equal only when all corresponding elements are equal

Therefore their size or dimensions are equal as well

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \qquad \mathbf{A} = \mathbf{B}$$

Matrices - Operations

ADDITION AND SUBTRACTION OF MATRICES

The sum or difference of two matrices, **A** and **B** of the same size yields a matrix **C** of the same size

$$c_{ij} = a_{ij} + b_{ij}$$

Matrices of different sizes cannot be added or subtracted

Matrices - Operations

SCALAR MULTIPLICATION OF MATRICES

Matrices can be multiplied by a scalar (constant or single element)

Let k be a scalar quantity; then

kA = Ak

Ex. If k=4 and

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 1 \\ 2 & -3 \\ 4 & 1 \end{bmatrix}$$

Matrices - Operations

MULTIPLICATION OF MATRICES

The product of two matrices is another matrix

Two matrices **A** and **B** must be **conformable** for multiplication to be possible

i.e. the number of columns of **A** must equal the number of rows of **B**

Example.

$$\mathbf{A} \quad \text{x} \quad \mathbf{B} \quad = \quad \mathbf{C}$$

$$(1x3) \quad (3x1) \quad (1x1)$$

Matrices - Operations

B x **A** = Not possible!

(2x1) (4x2)

A x **B** = Not possible!

(6x2) (6x3)

Example

A x **B** = **C**

(2x3) (3x2) (2x2)

Matrices - Operations

TRANSPOSE OF A MATRIX

If :

$$\underset{2x3}{A} = {}_2A^3 = \begin{bmatrix} 2 & 4 & 7 \\ 5 & 3 & 1 \end{bmatrix}$$

Then transpose of A, denoted A^T is:

$$A^T = {}_2A^{3^T} = \begin{bmatrix} 2 & 5 \\ 4 & 3 \\ 7 & 1 \end{bmatrix}$$

$$a_{ij} = a_{ji}^T$$ For all *i* and *j*

Matrices - Operations

INVERSE OF A MATRIX

Consider a scalar k. The inverse is the reciprocal or division of 1 by the scalar.

Example:

k=7 the inverse of k or $k^{-1} = 1/k = 1/7$

Division of matrices is not defined since there may be **AB** = **AC** while $\mathbf{B} \neq \mathbf{C}$

Instead matrix inversion is used.

The inverse of a square matrix, **A**, if it exists, is the unique matrix $\mathbf{A}^{-1}$ where:

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

Zero-One (Boolean) Matrix

Definition:

- Entries are Boolean values (0 and 1)
- Operations are also Boolean

Matrix *join*.

- $A \vee B = [a_{i,j} \vee b_{i,j}]$

Matrix *meet*.

- $A \wedge B = [a_{i,j} \wedge b_{i,j}]$

Example:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$A \vee B = \begin{bmatrix} 1\vee 0 & 0\vee 1 & 1\vee 0 \\ 0\vee 1 & 1\vee 1 & 0\vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$A \wedge B = \begin{bmatrix} 1\wedge 0 & 0\wedge 1 & 1\wedge 0 \\ 0\wedge 1 & 1\wedge 1 & 0\wedge 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Zero-One (Boolean) Matrix

Matrix multiplication: $A_{m \times k}$ and $B_{k \times n}$

- the **product** is a Zero-One matrix, denoted $A \circ B = C_{m \times n}$
- $c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2i}) \vee \ldots \vee (a_{ik} \wedge b_{kj})$.

Example:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad A \circ B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

...ual patterns. ... we simply ... the correspond...

Example 12 Let $\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

(a) Compute $\mathbf{A} \vee \mathbf{B}$. (b) Compute $\mathbf{A} \wedge \mathbf{B}$.

Solution

(a) Let $\mathbf{A} \vee \mathbf{B} = \left[c_{ij} \right]$. Then, since a_{43} and b_{43} are both 0, we see that $c_{43} = 0$. In all other cases, either a_{ij} or b_{ij} is 1, so c_{ij} is also 1. Thus

$$\mathbf{A} \vee \mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

(b) Let $\mathbf{A} \wedge \mathbf{B} = \left[d_{ij} \right]$. Then, since a_{11} and b_{11} are both 1, $d_{11} = 1$, and since a_{23} and b_{23} are both 1, $d_{23} = 1$. In all other cases, either a_{ij} or b_{ij} is 0, so $d_{ij} = 0$. Thus

$$\mathbf{A} \wedge \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

◆

Example 13 Let $\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$. Compute $\mathbf{A} \odot \mathbf{B}$.

Solution

Let $\mathbf{A} \odot \mathbf{B} = \left[e_{ij} \right]$. Then $e_{11} = 1$, since row 1 of **A** and column 1 of **B** each have a 1 as the first entry. Similarly, $e_{12} = 1$, since $a_{12} = 1$ and $b_{22} = 1$; that is, the first row of **A** and the second column of **B** have a 1 in the second position. In a similar way we see that $e_{13} = 1$. On the other hand, $e_{14} = 0$, since row 1 of **A** and column

4 of **B** do not have common 1's in any position. Proceeding in this way, we obtain

$$\mathbf{A} \odot \mathbf{B} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}.$$