Set and Set Operations

Introduction

- A **set** is a collection of objects.
- The objects in a set are called **elements** of the set.
- A **well – defined set** is a set in which we know for sure if an element belongs to that set.
- Example:
  - The set of all movies in which John Cazale appears is well – defined. (Name the movies, and what do they have in common? There are only 5.)
  - The set of all movie serials made by Republic Pictures is well – defined.
  - The set of best TV shows of all time is **not** well – defined. (It is a matter of opinion.)

Ways to define sets

- Explicitly: {John, Paul, George, Ringo}
- Implicitly: {1,2,3,…}, or {2,3,5,7,11,13,17,…}
- Set builder: { x : x is prime }, { x | x is odd }.
- In general { x : $P(x)$}, where $P(x)$ is some predicate.

We read
"the set of all x such that $P(x)$"

Set – Builder Notation

- When it is not convenient to list all the elements of a set, we use a notation the employs the rules in which an element is a member of the set. This is called set – builder notation.
- V = { people | citizens registered to vote in Maricopa County}
- $A = \{x \mid x > 5\}$ = This is the set A that has all real numbers greater than 5.
- The symbol | is read as such that.

Set properties 1

- Order does not matter
  - We often write them in order because it is easier for humans to understand it that way
  - {1, 2, 3, 4, 5} is equivalent to {3, 5, 2, 4, 1}

- Sets are notated with curly brackets

Set properties 2

- Sets do not have duplicate elements
  - Consider the set of vowels in the alphabet.
    - It makes no sense to list them as {a, a, a, e, i, o, o, o, o, o, u}
    - What we really want is just {a, e, i, o, u}
  - Consider the list of students in this class
    - Again, it does not make sense to list somebody twice
- Note that a list is like a set, but order does matter and duplicate elements are allowed
  - We won't be studying lists much in this class

Specifying a set 1

- Sets are usually represented by a capital letter (A, B, S, etc.)
- Elements are usually represented by an italic lower-case letter (*a*, *x*, *y*, etc.)
- Easiest way to specify a set is to list all the elements: A = {1, 2, 3, 4, 5}
  - Not always feasible for large or infinite sets

Specifying a set 2

- Can use an ellipsis (…): B = {0, 1, 2, 3, …}
  - Can cause confusion. Consider the set C = {3, 5, 7, …}. What comes next?
  - If the set is all odd integers greater than 2, it is 9
  - If the set is all prime numbers greater than 2, it is 11

- Can use set-builder notation
  - D = {*x* | *x* is prime and *x* > 2}
  - E = {*x* | *x* is odd and *x* > 2}
  - The vertical bar means "such that"
  - Thus, set D is read (in English) as: "all elements *x* such that *x* is prime and *x* is greater than 2"

Specifying a set 3

- A set is said to "contain" the various "members" or "elements" that make up the set
  - If an element *a* is a member of (or an element of) a set S, we use then notation $a \in S$
    - $4 \in \{1, 2, 3, 4\}$
  - If an element is not a member of (or an element of) a set S, we use the notation $a \notin S$
    - $7 \notin \{1, 2, 3, 4\}$
    - Virginia $\notin \{1, 2, 3, 4\}$

Special Sets of Numbers

- **N** = The set of natural numbers.

  = {1, 2, 3, …}.
- **W** = The set of whole numbers.

  ={0, 1, 2, 3, …}
- **Z** = The set of integers.

  = { …, -3, -2, -1, 0, 1, 2, 3, …}
- **Q** = The set of rational numbers.

  ={x| x=p/q, where p and q are elements of **Z and**

  q ≠ 0 }
- **H** = The set of irrational numbers.
- **R** = The set of real numbers.
- **C** = The set of complex numbers.

Universal Set and Subsets

- The **Universal Set** denoted by *U* is the set of all possible elements used in a problem.
- When every element of one set is also an element of another set, we say the first set is a **subset**.
- Example A={1, 2, 3, 4, 5} and B={2, 3}

  We say that B is a subset of A. The notation we use is $B \subseteq A$.
- Let S={1,2,3}, list all the subsets of S.
- The subsets of S are $\varnothing$, {1}, {2}, {3}, {1,2}, {1,3}, {2,3}, {1,2,3}.

The Empty Set

- The empty set is a special set. It contains no elements. It is usually denoted as { } or $\emptyset$.
- The empty set is always considered a subset of any set.
- Do not be confused by this question:
- Is this set {0} empty?
- It is not empty! It contains the element zero.

Intersection of sets

- When an element of a set belongs to two or more sets we say the sets will **intersect**.
- The intersection of a set A and a set B is denoted by $A \cap B$.
- $A \cap B = \{x | \text{ x is in A and x is in B}\}$
- Note the usage of and. This is similar to conjunction. A ^ B.
- Example A={1, 3, 5, 7, 9} and B={1, 2, 3, 4, 5}
- Then $A \cap B = \{1, 3, 5\}$. Note that 1, 3, 5 are in both A and B.

Mutually Exclusive Sets

- We say two sets A and B are mutually exclusive if $A \cap B = \varnothing$.
- Think of this as two events that can not happen at the same time.

Union of sets

- The union of two sets A, B is denoted by

  A U B.
- A U B = {x| x is in A or x is in B}
- Note the usage of or. This is similar to disjunction A v B.
- Using the set A and the set B from the previous slide, then the union of A, B is A U B = {1, 2, 3, 4, 5, 7, 9}.
- The elements of the union are in A or in B or in both. If elements are in both sets, we do not repeat them.

Complement of a Set

- The complement of set A is denoted by A' or by A^C.
- A' = {x| x is not in set A}.
- The complement set operation is analogous to the negation operation in logic.
- Example Say *U*={1,2,3,4,5}, A={1,2}, then A' = {3,4,5}.

Cardinal Number

- The **Cardinal Number** of a set is the number of elements in the set and is denoted by n(A).
- Let A={2,4,6,8,10}, then n(A)=5.
- The Cardinal Number formula for the union of two sets is

$$n(A \cup B)=n(A) + n(B) - n(A \cap B).$$

- The Cardinal number formula for the complement of a set is $n(A) + n(A')=n(U)$.

Examples.

{1, 2, 3} is the set containing “1” and “2” and “3.”

{1, 1, 2, 3, 3} = {1, 2, 3} since repetition is irrelevant.

{1, 2, 3} = {3, 2, 1} since sets are unordered.

{0,1, 2, 3, …} is a way we denote an infinite set (in this case, the natural numbers).

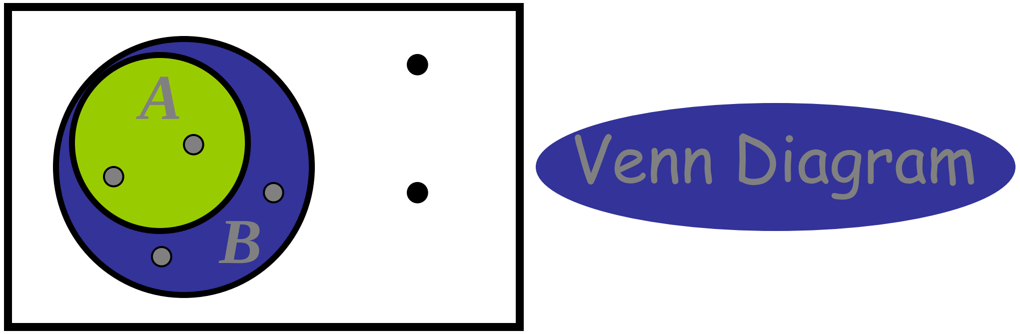
∅ = {} is the empty set, or the set containing no element

Note: ∅ ≠ {∅}

Definitions and notation

$x \in S$ means "x is an element of set S."
$x \notin S$ means "x is not an element of set S."
$A \subseteq B$ means "A is a subset of B."

or, "B contains A."
or, "every element of A is also in B."
or, $\forall x\ ((x \in A) \rightarrow (x \in B))$.



Definitions and notation

$A \subseteq B$ means "A is a subset of B."
$A \supseteq B$ means "A is a superset of B."

$A = B$ if and only if A and B have exactly the same elements

iff, $A \subseteq B$ and $B \subseteq A$
iff, $A \subseteq B$ and $A \supseteq B$
iff, $\forall x\, ((x \in A) \leftrightarrow (x \in B))$.

So to show equality of sets A and B, show:

$$A \subseteq B$$

$$B \subseteq A$$

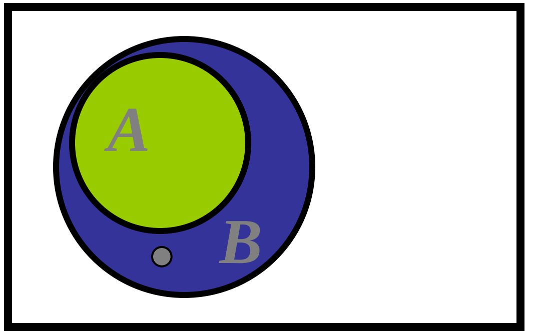
Definitions and notation

$A \subset B$ means "A is a proper subset of B."

– $A \subseteq B$, and $A \neq B$.

– $\forall x\ ((x \in A) \rightarrow (x \in B))$

$\wedge\ \exists x\ ((x \in B) \wedge (x \notin A))$



Definitions and notation

Quick examples:

- $\{1,2,3\} \subseteq \{1,2,3,4,5\}$
- $\{1,2,3\} \subset \{1,2,3,4,5\}$

Is $\varnothing \subseteq \{1,2,3\}$?

Yes! $\forall x\ (x \in \varnothing) \rightarrow (x \in \{1,2,3\})$ holds, because $(x \in \varnothing)$ is false.

Is $\varnothing \in \{1,2,3\}$? **No!**

Is $\varnothing \subseteq \{\varnothing,1,2,3\}$? **Yes!**

Is $\varnothing \in \{\varnothing,1,2,3\}$? **Yes!**

Cardinality

If S is finite, then the ***cardinality*** of S, $|S|$, is the number of distinct elements in S.

If $S = \{1,2,3\}$ — $|S| = 3.$

If $S = \{3,3,3,3,3\}$ — $|S| = 1.$

If $S = \varnothing$ — $|S| = 0.$

If $S = \{ \varnothing, \{\varnothing\}, \{\varnothing,\{\varnothing\}\} \}$ — $|S| = 3.$

If $S = \{0,1,2,3,\ldots\}$, $|S|$ is infinite. (more on this later)

Power sets

If S is a set, then the ***power set*** of S is

$$P(S) = 2^S = \{ x : x \subseteq S \}.$$

We say, "$P(S)$ is the set of all subsets of S."

If $S = \{a\}$ $\quad 2^S = \{\varnothing, \{a\}\}$.

If $S = \{a,b\}$ $\quad 2^S = \{\varnothing, \{a\}, \{b\}, \{a,b\}\}$.

If $S = \varnothing$ $\quad 2^S = \{\varnothing\}$.

If $S = \{\varnothing,\{\varnothing\}\}$ $\quad 2^S = \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing,\{\varnothing\}\}\}$.

Fact: if S is finite, $|2^S| = 2^{|S|}$. (if $|S| = n$, $|2^S| = 2^n$)

Cartesian Product

The ***Cartesian Product*** of two sets A and B is:

$$A \times B = \{ (a, b) : a \in A \wedge b \in B\}$$

If A = {Charlie, Lucy, Linus}, and
B = {Brown, VanPelt}, then

$A \times B$ = {(Charlie, Brown), (Lucy, Brown), (Linus, Brown), (Charlie, VanPelt), (Lucy, VanPelt), (Linus, VanPelt)}

We'll use these special sets soon!

$$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n): a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}$$

A,B finite $\rightarrow |A \times B| = |A||B|$

Set Operations

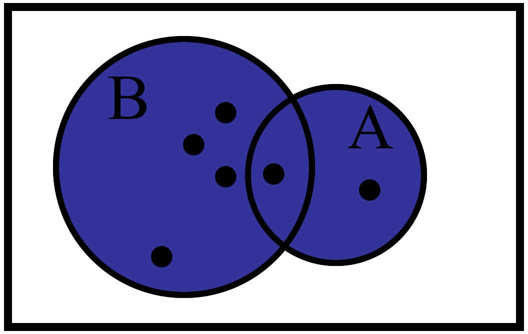
union

The *union* of two sets A and B is:

$$A \cup B = \{ x : x \in A \vee x \in B\}$$

If A = {Charlie, Lucy, Linus}, and
B = {Lucy, Desi}, then

$A \cup B$ = {Charlie, Lucy, Linus, Desi}

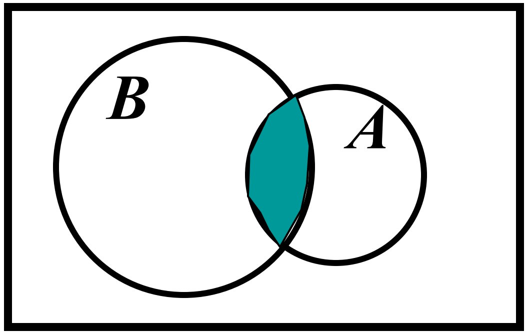


intersection

The *intersection* of two sets A and B is:

$$A \cap B = \{ x : x \in A \wedge x \in B \}$$

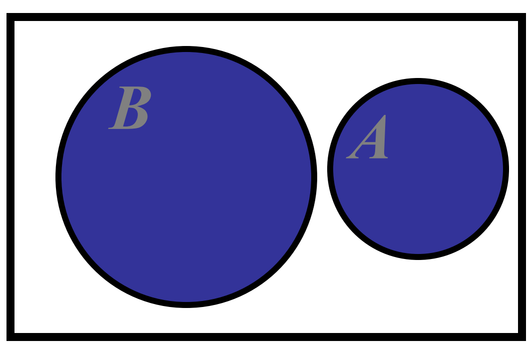
If A = {Charlie, Lucy, Linus}, and
B = {Lucy, Desi}, then
$A \cap B$ = {Lucy}



The *intersection* of two sets A and B is:

$$A \cap B = \{ x : x \in A \wedge x \in B \}$$

If $A = \{x : x$ is a US president$\}$, and
$B = \{x : x$ is in this room$\}$, then
$A \cap B = \{x : x$ is a US president in this room$\} = \emptyset$



Sets whose intersection is empty are called *disjoint* sets

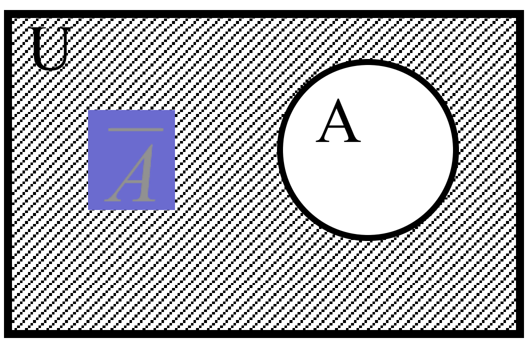
complement

The *complement* of a set *A* is:

$$\overline{A} = \{x : x \notin A\}$$

If $A = \{x : x \text{ is not shaded}\}$, then

$$\overline{A} = \{x : x \text{ is shaded}\}$$



$\overline{\varnothing} = U$

and

$\overline{U} = \varnothing$

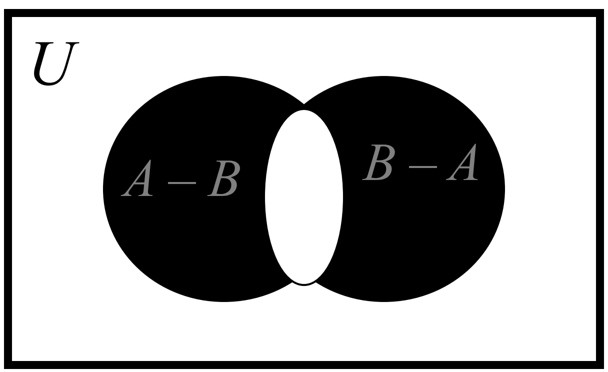
symmetric difference

The ***symmetric difference***, $A \oplus B$, is:

$$A \oplus B = \{\, x : (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A) \}$$

$$= (A - B) \cup (B - A)$$

$$= \{\, x : x \in A \oplus x \in B \}$$



Set Identities

- *Identity* $A \cap U = A$

  $A \cup \varnothing = A$

- ***Domination*** $A \cup U = U$

  $A \cap \varnothing = \varnothing$

- ***Idempotent*** $A \cup A = A$

  $A \cap A = A$

2.2 Set Identities

- *Excluded Middle* $A \cup \overline{A} = U$

- ***Uniqueness*** $A \cap \overline{A} = \varnothing$

- ***Double complement*** $\overline{\overline{A}} = A$

Set Identities

- *Commutativity* $A \cup B = B \cup A$

  $A \cap B = B \cap A$
- *Associativity* $(A \cup B) \cup C = A \cup (B \cup C)$

  $(A \cap B) \cap C = A \cap (B \cap C)$
- *Distributivity*

  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Subsets 1

- If all the elements of a set S are also elements of a set T, then S is a subset of T
  - For example, if S = {2, 4, 6} and T = {1, 2, 3, 4, 5, 6, 7}, then S is a subset of T
  - This is specified by $S \subseteq T$
    - Or by $\{2, 4, 6\} \subseteq \{1, 2, 3, 4, 5, 6, 7\}$
- If S is not a subset of T, it is written as such: $S \nsubseteq T$
  - For example, $\{1, 2, 8\} \nsubseteq \{1, 2, 3, 4, 5, 6, 7\}$

Subsets 2

- Note that any set is a subset of itself!
  - Given set S = {2, 4, 6}, since all the elements of S are elements of S, S is a subset of itself
  - This is kind of like saying 5 is less than or equal to 5
  - Thus, for any set S, $S \subseteq S$

Subsets 3

- The empty set is a subset of *all* sets (including itself!)
  - Recall that all sets are subsets of themselves
- *All* sets are subsets of the universal set
- A horrible way to define a subset:
  - $\forall x \; (x \in A \rightarrow x \in B)$
  - English translation: for all possible values of x, (meaning for all possible elements of a set), if x is an element of A, then x is an element of B
  - This type of notation will be gone over later

Proper Subsets 1

- If S is a subset of T, and S is not equal to T, then S is a proper subset of T
  - Let T = {0, 1, 2, 3, 4, 5}
  - If S = {1, 2, 3}, S is not equal to T, and S is a subset of T
  - A proper subset is written as $S \subset T$
  - Let R = {0, 1, 2, 3, 4, 5}. R is equal to T, and thus is a subset (but not a proper subset) or T
    - Can be written as: $R \subseteq T$ and $R \not\subset T$ (or just R = T)
  - Let Q = {4, 5, 6}. Q is neither a subset or T nor a proper subset of T

Proper Subsets 2

- The difference between “subset” and “proper subset” is like the difference between “less than or equal to” and “less than” for numbers

- The empty set is a proper subset of all sets other than the empty set (as it is equal to the empty set)

Proper subsets: Venn diagram

$S \subset R$